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EXTREMAL PROPERTIES OF STABLE RESONANCE MOTIONS*

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A system of differential equations, satisfying the phase volume preservation condition , is analyzed. It is shown that the Liapunov-stable resonance solutions of this system have extremal properties which can be elicited as functionals defined on the system's trajectory set.

The finding of stable resonance solutions is a part of the general problem of picking out the stable solutions of differential equation systems, formulated in /1/. A partial solution of it is given by an integral stability criterion /2/ valid for nonconservative mechanical systems whose motions are described by systems of equations containing a small parameter. This criterion emphasizes the extremal properties of the resonance solutions. On the basis of an extremal test introduced an attempt was made in /3/ to justify the evolution of the planets of the Solar system. Another approach to the problem of extremal properties of stable resonance motions is developed in /4/, where by example of a concrete mechanical system the possible validity of the following hypothesis was proved for a certain class of mechanical systems. Let $U(q, t)$ be a force function depending on the generalized coordinate vector q and periodic in the independent argument t with period T and let $q(q_0, q_0^{\cdot}, t)$ be the solution of the system of equations of motion with initial data $g_0, q_0^{\cdot}$ at instant $t = 0$. It is assumed that the function

$$\langle U(q_0, q_0^{\cdot})\rangle = \lim_{m\to\infty} \frac{1}{mT} \int_0^{mT} U(q(q_0, q_0^{\cdot}, t), t)\, dt$$

(m is an integer) reaches maximum values on the set of initial states at the points $q_*, q_*^{\cdot}$ corresponding to the initial values of the Liapunov-stable (with respect to variables $q, q^{\cdot}$) resonance motions. An expanded proof of the theorem stated in /5/ is given below, completing certain gaps in the hypothesis presented.

1. We consider the nonautonomous periodic system

$$dx/dt = X(x, t),\ \ X(x, t+1) \equiv X(x, t),\ \ x \in R^n,\ \ \operatorname{div} X \equiv 0 \tag{1.1}$$

The last condition (preservation of phase volume) is satisfied, for example, by canonic systems. Let $G_0 \subset R^n$ be the set of initial state, defined at the instant $t = 0$ and g_0^t be a transormation prescribed by system (1.1) for $t \geqslant 0$, which takes the system from the initial point $x \in G_0$ to a point $x_t \in R^n$, i.e., $x_t = g_0^t x$ is the solution of system (1.1) with the initial value x. It is assumed that the solutions satisfy the conditions of uniqueness and of continuous dependence on the initial data.

Theorem. In order that system (1.1) admit of a Liapunov-stable periodic solution of a period that is a multiple of unity, it is necessary and sufficient that the following conditions be fulfilled:

1) the system admits of an open set $A_0 \subset G_0$, $\operatorname{mes}(A_0) \neq 0$, such that the set $\{x_t, t \geqslant 0, x \in A_0\}$ of semitrajectories is embedded in some compactum M (a compact of Hausdorff space);

2) a function $\varkappa(x, t)$ from the set of functions continuous in x on M and continuous and periodic in t with a period commensurable with unity exists such that the function

$$K(x) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \varkappa(x_t, t)\, dt \tag{1.2}$$

is continuous at point $y \in A_0$ and takes a strict extremal value (minimum or maximum) at this

*Prikl.Matem.Mekhan.,45,No.6,1009-1015,1981 point; the point y is the initial value of the desired periodic solution.

The existence of function $K(x)$ is guaranteed when condition 1 is fulfilled for almost all $x \in A_0$. This follows from the existence of a compactum $B \subset M$, $\operatorname{mes}(B) \neq 0$, invariant relative to $g_0{}^m$-transformations (m belongs to the integer set Z). Indeed, consider an open measurable set $B^\circ = \bigcup_{k=0}^{\infty} A_k$, where $A_k = g_0{}^k A_0$, $k \in Z$. Obviously, $g_0{}^1 B^\circ \subset B^\circ$. From the $g_0{}^1$-transformation's property of phase volume preservation we conclude that $g_0{}^1 B^\circ = B^\circ$ and, consequently, $g_0{}^1 B = B$, where $B = \overline{B^\circ}$. The conclusion on the existence of function $K(x)$ for almost all $x \in B$ can be obtained by extending the proof of the ergodic theorem for autonomous systems /6/ to the case of periodic systems (1.1) admitting of an invariant compactum B (relative to $g_0{}^m$-transformations).

Let $\rho(x, y)$ be the Euclidean metrics of the phase space R^n, $S(\varepsilon, y) = \{x: \rho(x, y) < \varepsilon\}$ be an open sphere of radius ε centered at point y. By the continuity of function $K(x)$ at point y we mean continuity in the sense of the topology induced of the definition set of this function by the topology of space R^n. The condition of strict minimum of function $K(x)$ at point y implies the following: a sphere $S(R, y) \subset A_0$ exist such that

$$\inf_{x \in D(R,\varepsilon)} |K(x) - K(y)| > 0, \ D(R, \varepsilon) = S'(R, y) \setminus S'(\varepsilon, y)$$

for any ε from the interval $(0, R)$. The prime denotes the set's points at which the function $K(x)$ is defined ($S' \subset S$, $\operatorname{mes}(S') = \operatorname{mes}(S)$).

Let us now prove the theorem.

Sufficiency. Without loss of generality we can take it that the function $\varkappa(x, t)$ is periodic with unit period and furnishes the function $K(x)$ with a minimum equal to zero. We consider the trajectory leaving the point x_n, $n \in Z$, at instant $t = 0$. From (1.2) follows

$$K(x_n) = K(x) \tag{1.3}$$

We fix an arbitrary $\varepsilon \in (0, R)$ and we find

$$a = \inf_{x \in D(R, \varepsilon)} K(x) > 0, \quad K(x) \geqslant a, \ x \in D(R, \varepsilon) \tag{1.4}$$

As a consequence of the continuity of $K(x)$ at point y there exists $\delta \in (0, \varepsilon)$ such that

$$K(x) < a, \ x \in S'(\delta, y) \tag{1.5}$$

We denote the sets $S(R, y)$ and $S(\delta, y)$ by α_0 and β_0 and we consider the sets $\alpha_n = g_0{}^n \alpha_0$, $\beta_n = g_0{}^n \beta_0$. We have

$$\beta_n \subset \alpha_n, \ \operatorname{mes}(\alpha_n) = \operatorname{mes}(\alpha_0) > \operatorname{mes}(\beta_n) = \operatorname{mes}(\beta_0) > 0 \tag{1.6}$$

By virtue of the properties of the $g_0{}^n$-transformation and of equality (1.3) the inequalities (1.4), (1.5) are preserved if $D(R, \varepsilon)$ and $S'(\delta, y)$ are replaced by $\alpha_n' \setminus \beta_n'$ and β_n', respectively.

Let us show that y is the initial value of a periodic solution of a period cummensurable with unity. For this it is sufficient that the condition $y_N \in \alpha_0$ be fulfilled for some positive integer N (then from the theorem's condition 2 it follows that $y_N = y$). If we assume the contrary, then from (1.3)–(1.6) it follows that for any $n, m \in Z$, $n \neq m$, the sets α_n, α_m either do not intersect or their intersection lies in the set $\alpha_n \setminus \beta_n$. In both cases $\beta_n \setminus \beta_m = \varnothing$. Hence it follows that a countable number of nonintersecting sets β_n of one and the same nonzero measure have been imbedded in compactum B, which cannot be. The contradiction obtained proves the existence of an N for which $y_N = y$ is fulfilled, i.e., y is the initial value of a periodic solution.

Let us show the Liapunov stability (with respect to variable x) of this solution. We take an arbitrary $\varepsilon \in (0, R)$. As a consequence of the continuity of the solution with respect to the initial data there exists $\varepsilon_1 \in (0, \varepsilon)$ such that

$$x_t \in S(\varepsilon, y_t), \ x \in S(\varepsilon_1, y), \ t \in [0, N] \tag{1.7}$$

Analogously to how conditions (1.4), (1.5) were obtained, we derive

$$\begin{gathered} K(x) \geqslant a, \ x \in D(\varepsilon, \varepsilon_1); \ K(x) < a, \ x \in S'(\delta, y) \\ a = \inf_{x \in D(\varepsilon, \varepsilon_1)} K(x), \ D(\varepsilon, \varepsilon_1) = S'(\varepsilon, y) \setminus S'(\varepsilon_1, y), \ \delta \in (0, \varepsilon_1) \end{gathered} \tag{1.8}$$

Allowing for (1.3), (1.8) and the fact that y is a fixed point of the g_0^{Nk}-transformation when $k \in Z$, we obtain

$$g_0^{Nk}(S'(\delta, y)) \subset S(\varepsilon_1, y) \tag{1.9}$$

From the properties of the $g_0{}^t$-transformation follows

$$g_0^{Nk}(S(\delta, y)) \subset S(\varepsilon_1, y) \tag{1.10}$$

Combining (1.7) and (1.10) we conclude that the solution $\{y_t, t \geqslant 0\}$ is stable.

Necessity. The necessity of fulfilling the theorem's condition 1 in the case when system (1.1) admits of a stable periodic solution $\{y_t, t \geqslant 0\}$ with a period commensurable with unity is obvious. In order to satisfy condition 2 we consider the function $\varkappa(x, t) = \rho(x, y_t)$. The function

$$K(x) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \rho(x_t, y_t)\, dt$$

determines the mean distance between the trajectories with initial values x and y. It can be shown that function $K(x)$ is continuous at point y and at it takes a minimum value equal to zero. We fix an arbitrary $\varepsilon_1 > 0$ and we find $\varepsilon_2 \in (0, \varepsilon_1)$ such that $\rho(x_t, y_t) < \varepsilon_1$ for $t \geqslant 0$, $x \in S(\varepsilon_2, y)$. We consider an arbitrary $\varepsilon \in (0, \varepsilon_2)$. Let us show that any trajectory starting from set $S(\varepsilon_2, y) \setminus S(\varepsilon, y)$ at instant $t = 0$ cannot approach arbitrarily closely the periodic trajectory being investigated at the succeeding instants $t > 0$. It is well known that any ε-neighborhood of point y contains a compactum B_ε invariant relative to g_0^{Nk}-transformations (N is the minimal whole period of solution y_t, $k \in Z$), and the point y, fixed under these transformations, is an internal point of B_ε /7/. Let d_t be the precise lower bound of the distance of point y_t from the boundary of set $g_0{}^tB_\varepsilon$. Obviously, $d_t > 0$ for any finite instant t. The equality $\inf_{t\geqslant 0} d_t = \inf_{t\in[0,N]} d_t$ completes the proof since $x_t \notin g_0{}^tB_\varepsilon$ for $x \in \overline{D(\varepsilon, \varepsilon_2)}$.

Corollary. The equilibrium position $\{y_t \equiv y, t \geqslant 0\}$ of an autonomous or nonautonomous system (1.1) is stable if and only if condition 2 is fulfilled.

Together with the periodic solutions of system (1.1) we have considered we can have resonance solutions of the form

$$x = \omega t + z; \quad x, z, \omega \in R^n \tag{1.11}$$

where each coordinate ω^i of the vector ω is commensurable with 2π, while $z = z(t)$ is a vector periodic in t, admitting of a period that is a multiple of unity. The replacement of variables x by z in accord with formulas (1.11) reduces the finding of the resonance solution (1.11) to the search for a periodic solution $z(t)$ of a new system. If after the replacement indicated the right-hand side of system (1.1) retains the property of periodicity with respect to the variable t occurring explicitly in the equation, then the stable periodic solution $z(t)$, and, consequently, the stable resonance solution (1.11) can be elicited by the extremal properties of the mean values (1.2) of certain functions $\varkappa(x, t)$. Such a case arises, for example, if system (1.1) is (2π)-periodic in each of the rotational coordinates x^i $(\omega^i \neq 0)$.

The theorem presented can be considered a justification and a generalization of an idea suggested in /4/. Resonance planar rotations of a satellite around the center of mass in a gravitational field as the satellite moved on an unperturbed Keplerian orbit were investigated in /4/. The satellite's rotation is described by a differential equations system of type (1.1), (2π)-periodic in the independent variable t; $x = (\delta, \delta^{\cdot})$, $\delta = 2\theta$, where θ is the angle of deviation of the satellite's axis from the orbit's radius-vector, $\delta^{\cdot} = d\delta/dt$. The solutions with the rotational coordinate

$$\delta = \pm (p/q)\, t + \psi(t); \quad p, q \in Z \tag{1.12}$$

are the resonance rotations ($\psi(t)$ is a function periodic in t, with a period a multiple of $2\pi q$). According to the hypothesis and the numerical calculations the maxima of the function

$$K(\delta_0, \delta_0^{\cdot}) = \lim_{t\to\infty} \frac{1}{t} \int_0^t U(\delta(\delta_0, \delta_0^{\cdot}, t), t)\, dt$$

yield the initial values of the stable resonance solutions; $U(\delta, t)$ is the problem's force function, (2π)-periodic in t and δ. Since in this case system (1.1) is (2π)-periodic in δ, the stable solution (1.12) satisfies the theorem, and the function U is one of the admissible $\varkappa(x, t)$-functions. Consequently, the theorem obtained gives a theoretical confirmation of the hypothesis. Together with this we stress that the hypothesis has a global character, whereas from the theorem we should conclude that to each stable resonance motion there corresponds, in general, its own function $\varkappa(x, t)$.

2. As a possible application of the theorem we consider the problem of the perturbed motion of a rigid body in a gravitational field around the center of mass, moving along a perturbed orbit having a constant inclination to the ecliptic and rotating uniformly around the ecliptic's axis with a constant angular velocity. The equations of perturbed motion can be reduced to a canonic form with the Hamiltonain $H^*(x, \tau)$. Here $x = (L, L_n, l, \psi, \sigma, \varphi)$ are the problem's evolution variables, τ is the independent variable. H^* is a (2π)-periodic function of $\tau, \psi, \sigma, \varphi$ /8,9/. Consequently, the stable resonance solutions of form (1.11) with on rotational coordinate

$$\Psi = (1/2)\, n\tau + \delta$$

where n is an integer and $\delta = \delta(\tau)$ is a periodic function of τ, admitting of a period a multiple of 2π, satisfy the theorem presented.

Let us consider the concrete value $n = n_0$ and carry out the replacement (1.11) of vector $x = (L, L_n, l, \psi, \sigma, \varphi)$ by $z = (L, L_n, l, \delta, \sigma, \varphi)$, by the same token reducing the problem to a search for the stable periodic solution $z(\tau)$ of a new equation system which too can be written in a canonic form with the Hamiltonian

$$H(z, \tau) = H^*_{|x \to z} - (1/2)\, n_0 L$$

We denote the period of H with respect to τ by T_τ. The extremum of function

$$K(z_0) = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau H(z_\tau, \tau)\, d\tau, \quad z_0 = z(0)$$

(if there is one) determines the existence and stability of the required periodic motion, when condition 1 is fulfilled. It is essential that the function $K(z_0)$ be determined along the exact trajectories which are unknown but can be obtained approximately, for example, by numerical methods.

Using the fact that under certain assumptions z is the vector of slow variables, we compute the function $K(z_0)$ approximately, along the trajectories of the unperturbed problem, which is easily integrated $(z = z_0)$:

$$\langle H \rangle = \frac{1}{T_\tau} \int_0^{T_\tau} H(z_0, \tau)\, d\tau$$

To the extrema of function $\langle H \rangle$ correspond the Cassini-motions, namely, stable resonance solutions of the averaged equations of motion. It is probable that the existence and stability of "the generalized Cassini's laws" are an approximate description of the conditions of existence and stability of the resonance solutions of the unaveraged system, furnished by the extrema of function $K(z_0)$. We remark that the existence of such solutions has been proved by the small parameter method /10/.

3. Let system (1.1) have a periodic solution $\{y_t, t \geqslant 0\}$ whose stability can be established by the construction of an appropriate Liapunov function $V(x, t)$ in the form of a bundle of first integrals of the system. Since $V = \text{const}$ on the trajectories of system (1.1), the function

$$K(x) = \lim_{t\to\infty} \frac{1}{t} \int_0^t V(x_t, t)\, dt = V(x, 0)$$

takes a strict extremum at point y, corresponding to a resonance solution. Consequently, as such a Liapunov function we can take one of the $\varkappa(x, t)$-functions exhibiting the extremal character of the stable resonance solution.

For example, if we are examining the case of a circular orbit of the satellite's center of mass in the problem in /4/, then the satellite's planar rotation is described by an autonomous canonic first-order system with a Hamiltonian $H(q, p)$, admitting of the steady-state solution $q = p = 0$ in whose neighborhood the Hamiltonian H is a positive-definite function. Since the equation system has a first integral $H = \text{const}$, the steady-state solution is stable on the strength of Liapunov's theory. Consider the function $\varkappa = H(q, p)$; we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t H(q(t), p(t))\,dt = H(q(0), p(0))$$

On the strength of the theorem's corollary we have obtained a new proof of the stability of the resonance solution $q = p = 0$ with an obvious extremal content. For noncircular orbits the Hamiltonian $H(q, p, t)$ of the nonautonomous problem in /4/ is not a first integral, but the function

$$K(q(0), p(0)) = \lim_{t\to\infty}\frac{1}{t}\int_0^t H(q(t), p(t), t)\,d$$

can elicit the initial data of the stable resonance motions.

4. Consider the autonomous equation system

$$dx/dt = X(x),\ x \in R^n,\ \operatorname{div} X \equiv 0 \qquad (4.1)$$

Let the system be 2π-periodic in the variables $x^1, \ldots, x^k$ and a periodic in the remaining variables $x^{k+1}, \ldots, x^n$. From the phase space R^n we pass to the phase cylinder

$$\Omega = [x^1]_{\mathrm{mod}\,2\pi} \times \ldots \times [x^k]_{\mathrm{mod}\,2\pi} \times R^{n-k}$$

We assume that on the phase cylinder there exists a compactum M of nonzero measure, invariant relative to any g_0^t-transformations. The extremal properties of the stable resonance solutions of system (4.1) are established by the theorem proved, if we replace the set of functions $\varkappa(x, t)$ examined in the nonautonomous case of (1.1) by a set of functions continuous in x on M and periodic in t (without restrictions on the value of the period).

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